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Probabilistic Sophistication, Second Order Stochastic Dominance, and Uncertainty Aversion*

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Abstract

We study the interplay of probabilistic sophistication, second order stochastic dominance, and uncertainty aversion, three fundamental notions in choice under uncertainty. In particular, our main result, Theorem 2, characterizes uncertainty averse preferences that satisfy second order stochastic dominance, as well as uncertainty averse preferences that are probabilistically sophisticated.

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1 Introduction

In this paper we study in depth the overlap between the two fundamental classes of probabilistically sophisticated preferences and uncertainty averse preferences. The former class of preferences, introduced and axiomatized by Machina and Schmeidler [15], rank acts $f : S \rightarrow X$ according to

$$V(f) = M(q \circ f^{-1}) \tag{1}$$

where q is a reference probability measure, $q \circ f^{-1}$ is the lottery induced on X by f under q , and M is a functional over the set of lotteries on X . Probabilistically sophisticated preferences characterize decision makers that are able to quantify their beliefs with a single probability measure q , but that on the induced lotteries $q \circ f^{-1}$ do not necessarily satisfy the expected utility axioms.¹

Uncertainty averse preferences are complete and transitive preferences that are both monotone and convex. It is a very large class of preferences, arguably the most basic class of rational preferences that exhibit a negative attitude toward uncertainty. Recently, in Cerreia-Vioglio, Maccheroni, Marinacci,

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¹Papers that study the properties of probabilistically sophisticated preferences include Grant [11], Machina and Schmeidler [16], Sarin and Wakker [20], Grant and Polak [12], Chew and Sagi [5] and [6], and Kopylov [13].

and Montrucchio [2] we establish a representation for uncertainty averse preferences that plays a key role in our analysis. Specifically, we show that these preferences rank acts according to

$$V(f) = \min_{p \in \Delta} G \left(\int u(f) dp, p \right) \quad (2)$$

where $G : \mathbb{R} \times \Delta \rightarrow (-\infty, \infty]$ is a quasiconvex function that is increasing in the first component, and $u : X \rightarrow \mathbb{R}$ is an affine function. The function G is an index of uncertainty aversion, while u captures risk aversion.

In view of the representation (2), to study the overlap between probabilistically sophisticated and uncertainty averse preferences amounts to determine what properties of the index G characterize uncertainty averse preferences that are probabilistically sophisticated. Our main result, Theorem 2, achieves this goal by showing, under a standard assumption of nonatomicity, that a suitable symmetry property of G , called rearrangement invariance, characterizes the uncertainty averse preferences that are probabilistically sophisticated. In this way, we considerably extend earlier results of Maccheroni, Marinacci, and Rustichini [18] for variational preferences, a special class of uncertainty averse preferences.

Theorem 2 also establishes some remarkable properties of uncertainty averse preferences that satisfy second order stochastic dominance, a property widely used in economic applications. As discussed after its statement, in a sense Theorem 2 can be viewed as, first of all, a result on the overlap between uncertainty aversion and second order stochastic dominance.

As a byproduct of Theorem 2, in Proposition 3 we show that in the presence of a nontrivial unambiguous event the overlap collapses to the class of subjective expected utility preferences. This shows that the basic tension first identified by Marinacci [17] among probabilistic sophistication and the multiple priors representation (recently extended by Strzalecki [23] to variational preferences) extends much more generally to the uncertainty averse case. Moreover, Proposition 3 shows that this tension is actually peculiar to second order stochastic dominance, which turns out to be the general property with a problematic interplay with uncertainty aversion.

Mathematically, our results build on the theory of rearrangement invariant Banach spaces, first studied in the seminal paper of Luxemburg [14]. More precisely, Theorem 2 depends on a dual characterization of quasiconcave and rearrangement invariant functionals defined over the normed space of simple functions. This characterization shares some of the techniques of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [4], where quasiconvex and rearrangement invariant functionals defined over $L^\infty(S, \Sigma, q)$ are studied and characterized. However, the present very different decision theoretic setting makes the derivation quite delicate and different. All this is reported in Appendix, where all proofs are collected.

2 Preliminaries

2.1 Mathematical Preliminaries

We consider an Anscombe-Aumann setup [1]. Let S be a state space endowed with an event σ -algebra Σ , and X a convex set of consequences. We denote by \mathcal{F} the set of all simple acts $f : S \rightarrow X$, that is, the set of all Σ -measurable maps that take on finitely many values. Given $A \in \Sigma$ and $f, g \in \mathcal{F}$, we denote by gAf the simple act that yields $g(s)$ if $s \in A$ and $f(s)$ if $s \notin A$.

Let $B_0 = B_0(S, \Sigma)$ be the set of simple Σ -measurable functions, $\varphi : S \rightarrow \mathbb{R}$, endowed with the supnorm. Denote by Δ the set of all finitely additive probabilities on Σ , endowed with the weak* topology. The subset of Δ consisting of all countably additive probabilities on Σ is denoted by Δ^σ . Given $q \in \Delta^\sigma$, denote by $\Delta^\sigma(q) = \{p \in \Delta : p \ll q\}$ the set of all countably additive probabilities on Σ that are absolutely continuous with respect to (wrt, for short) q . Finally, when $q \in \Delta^\sigma$, we say that (S, Σ, q) is *adequate* if either q is nonatomic or if S is finite and q is uniform.

Endow $\mathbb{R} \times \Delta$ with the product topology and define $\mathcal{L}(\mathbb{R} \times \Delta)$ as the class of functions $G : \mathbb{R} \times \Delta \rightarrow (-\infty, \infty]$ such that:

- (i) $G(\cdot, p)$ is an increasing function for all $p \in \Delta$;
- (ii) G is quasiconvex and lower semicontinuous;
- (iii) $\min_{p \in \Delta} G(t, p) = t$ for all $t \in \mathbb{R}$.

A function $G \in \mathcal{L}(\mathbb{R} \times \Delta)$ is *linearly continuous* if the function $I : B_0 \rightarrow \mathbb{R}$ defined by

$$I(\varphi) = \min_{p \in \Delta} G\left(\int \varphi dp, p\right) \quad (3)$$

is continuous. For example, [2] shows that $G \in \mathcal{L}(\mathbb{R} \times \Delta)$ is linearly continuous if $G(\cdot, p)$ is upper semicontinuous on \mathbb{R} for each $p \in \Delta$.

2.2 Decision Theoretic Setup

We consider a binary relation \succsim on \mathcal{F} that satisfies the following classic axioms:

A 1 (Weak Order) *The binary relation \succsim is nontrivial, complete, and transitive.*

A 2 (Monotonicity) *If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.*

A 3 (Uncertainty Aversion) *If $f, g \in \mathcal{F}$ and $\alpha \in (0, 1)$, $f \sim g$ implies $\alpha f + (1 - \alpha)g \succsim f$.*

Following [2], a preference relation \succsim that satisfies axioms A1-A3 is called *uncertainty averse*. As [2] argues at length, this is the most basic class of rational preferences on \mathcal{F} that exhibit a negative attitude toward uncertainty.

To derive a representation for uncertainty averse preferences we need some further mild axioms. The following axiom is peculiar to the Anscombe-Aumann setting and is a standard independence axiom on constant acts, that is, on acts that only involve risk and no state uncertainty.

A 4 (Risk Independence) *If $x, y, z \in X$ and $\alpha \in (0, 1)$, $x \sim y$ implies $\alpha x + (1 - \alpha)z \sim \alpha y + (1 - \alpha)z$.*

The next axioms are technical conditions that simplify the derivation and make the representation more tractable.

A 5 (Continuity) *If $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0, 1] : \alpha f + (1 - \alpha)g \succsim h\}$ and $\{\alpha \in [0, 1] : h \succsim \alpha f + (1 - \alpha)g\}$ are closed.*

A 6 (Unboundedness) *There are $x, y \in X$ such that, for each $\alpha \in (0, 1)$, there exist $z, z' \in X$ such that $\alpha z + (1 - \alpha)y \succ x \succ y \succ \alpha z' + (1 - \alpha)x$.*

A 7 (Monotone Continuity) *If $f, g \in \mathcal{F}$, $x \in X$, $\{E_n\}_n \in \Sigma$ with $E_n \downarrow \emptyset$, then $f \succ g$ implies that there exists $n_0 \in \mathbb{N}$ such that $xE_{n_0}f \succ g$.*

If \succsim satisfies axioms A1, A2, and A5, then each act $f \in \mathcal{F}$ has a certainty equivalent $x_f \in X$; i.e., $f \sim x_f$. Certainty equivalents play an important role in the following representation result for uncertainty averse preferences, proved in [2]. Here $\mathcal{U}(X)$ is the class of affine functions $u : X \rightarrow \mathbb{R}$.

Theorem 1 *Let \succsim be a binary relation on \mathcal{F} . Then, the following conditions are equivalent:*

(i) \succsim satisfies axioms A1-A7;

(ii) *there exist $u \in \mathcal{U}(X)$, with $u(X) = \mathbb{R}$, and $G \in \mathcal{L}(\mathbb{R} \times \Delta)$ linearly continuous with $\text{dom } G \subseteq \mathbb{R} \times \Delta^\sigma$,² such that, for each f and g in \mathcal{F} ,*

$$f \succsim g \iff \min_{p \in \Delta} G\left(\int u(f) dp, p\right) \geq \min_{p \in \Delta} G\left(\int u(g) dp, p\right). \quad (4)$$

The function u is cardinally unique and, given u , the unique $G \in \mathcal{L}(\mathbb{R} \times \Delta)$ that satisfies (4) is

$$G(t, p) = \sup_{f \in \mathcal{F}} \left\{ u(x_f) : \int u(f) dp \leq t \right\}. \quad (5)$$

Observe that the technical axioms A5-A7 translate in the representation as follows: A5 guarantees the linear continuity of G , A6 corresponds to $u(X) = \mathbb{R}$, and A7 implies that $\text{dom } G \subseteq \mathbb{R} \times \Delta^\sigma$.³

Theorem 1 motivates the following definition.

Definition 1 *A pair $(u, G) \in \mathcal{U}(X) \times \mathcal{L}(\mathbb{R} \times \Delta)$ that represents a binary relation \succsim in the sense of point (ii) of Theorem 1 is called an uncertainty averse representation of \succsim .⁴*

Behaviorally, by Theorem 1 a binary relation admits an uncertainty averse representation if and only if it satisfies axioms A1-A7. As shown in [2, Proposition 12], the variational preferences of [18] correspond to the additively separable case

$$G(t, p) = t + c(p) \quad (6)$$

where $c : \Delta \rightarrow [0, \infty]$ is a lower semicontinuous convex function.

The function G is an index of uncertainty aversion. Given two preferences \succsim_1 and \succsim_2 , based on Ghirardato and Marinacci [8] say that \succsim_1 is *more uncertainty averse* than \succsim_2 if, for each $f \in \mathcal{F}$ and each $x \in X$,

$$f \succsim_1 x \implies f \succsim_2 x. \quad (7)$$

In [2, Proposition 6] we prove that, if \succsim_1 and \succsim_2 have uncertainty averse representations (u_1, G_1) and (u_2, G_2) , the following conditions are equivalent:

²Recall that $\text{dom } G = \{(t, p) \in \mathbb{R} \times \Delta : G(t, p) < \infty\}$.

³[2, Theorem 3] provides a more general representation result that does not rely on A6 and A7.

⁴Though to ease terminology we use the term *uncertainty averse representation*, to be precise we should have used the term *surjective and monotone continuous, uncertainty averse representation*.

- (i) \succsim_1 is more uncertainty averse than \succsim_2 ,
- (ii) u_1 is cardinally equivalent to u_2 and, normalizing $u_1 = u_2$, $G_1 \leq G_2$.

We close with a representation result that shows what the classic axioms of Savage [21] imply in the present Anscombe-Aumann setting. It is an essentially known result, studied for example by Neilson [19] and, more recently, by Strzalecki [22]. For completeness, in the Appendix we report a proof of this result since it will play an important role in what follows and we did not find a proof of the version that we use.

Proposition 1 *Let \succsim be a binary relation on \mathcal{F} . Then, the following conditions are equivalent:*

- (i) \succsim satisfies Savage's axioms P1-P6 and axioms A4-A5;
- (ii) there exist a nonatomic probability measure q , a nonconstant affine $u : X \rightarrow \mathbb{R}$, and a strictly increasing and continuous function $\phi : u(X) \rightarrow \mathbb{R}$ such that, for each f and g in \mathcal{F} ,

$$f \succsim g \iff \int \phi(u(f)) dq \geq \int \phi(u(g)) dq. \quad (8)$$

The probability q is unique, u is cardinally unique, and ϕ is cardinally unique given u .⁵ Moreover, ϕ is concave if and only if \succsim satisfies A3, and $q \in \Delta^\sigma$ if and only if \succsim satisfies A7.

In other words, $\int \phi(u(f)) dq$ represents a preference \succsim of a Savage decision maker in an Anscombe-Aumann world.

3 Main Result

In this section we state the paper's main result, Theorem 2, which characterizes uncertainty averse preferences that are probabilistically sophisticated.

Fix a reference probability $q \in \Delta^\sigma$. In [18, Theorem 14] it is shown that variational preferences are probabilistically sophisticated wrt q if and only if the uncertainty aversion index c in (6) is rearrangement invariant wrt q , provided (S, Σ, q) is adequate.⁶ Hence, probabilistic sophistication translates into a property of symmetry of the index c when preferences are variational. Here, Theorem 2 shows that this property can be suitably generalized to all uncertainty averse preferences.

To state the main result we introduce few classic notions, key for our analysis. A preference relation \succsim on \mathcal{F} :

- (i) is *probabilistically sophisticated* (wrt q) if, given any $f, g \in \mathcal{F}$,

$$q(\{s \in S : f(s) = x\}) = q(\{s \in S : g(s) = x\}) \quad \forall x \in X \implies f \sim g; \quad (9)$$

- (ii) satisfies *first order stochastic dominance* (wrt q) if, given any $f, g \in \mathcal{F}$,

$$q(\{s \in S : f(s) \preceq x\}) \leq q(\{s \in S : g(s) \preceq x\}) \quad \forall x \in X \implies f \succsim g, \quad (10)$$

or, equivalently,

$$\int \phi(u(f)) dq \geq \int \phi(u(g)) dq \quad \forall \phi \in \Phi_{mi} \implies f \succsim g, \quad (11)$$

⁵See (21) in Appendix and its discussion for more details on uniqueness.

⁶Rearrangement invariance is a symmetry property that will be introduced shortly.

where Φ_{mi} is the set of all increasing functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$,⁷

(iii) satisfies *second order stochastic dominance* (wrt q) if, given any $f, g \in \mathcal{F}$,

$$\int \phi(u(f)) dq \geq \int \phi(u(g)) dq \quad \forall \phi \in \Phi_{icv} \implies f \succsim g, \quad (12)$$

where Φ_{icv} is the set of all concave and increasing functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$.⁸

To interpret the latter dominance conditions, consider decision makers that rank acts in \mathcal{F} according to the representation (8). A preference \succsim satisfies second order stochastic dominance when it preserves the unanimous preference of all these decision makers that are uncertainty averse (i.e., that feature a concave ϕ) and share the same u over the constant acts X . Analogously, \succsim satisfies first order stochastic dominance when it preserves the unanimous preference of all monotone decision makers (i.e., that feature an increasing ϕ) and share the same u over X .

These dominance notions can be also easily interpreted in terms of the classic notions of stochastic dominance on lotteries. In fact, (11) is equivalent to require that the lottery induced by $u \circ f$ under q first order stochastically dominates that induced by $u \circ g$, while (12) is equivalent to require that the former lottery second order stochastically dominates the latter one.

Our results use the convex order, a classic stochastic order. Specifically, the *convex order* \succsim_{cx} on $L^1(q) = L^1(S, \Sigma, q)$ is defined by

$$\varphi \succsim_{cx} \psi \iff \int \ell(\varphi) dq \geq \int \ell(\psi) dq \quad \forall \ell \in \Phi_{cx}, \quad (13)$$

where Φ_{cx} is the set of all convex functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$. Notice that this order can be also defined over $\Delta^\sigma(q)$ by

$$p \succsim_{cx} p' \iff \frac{dp}{dq} \succsim_{cx} \frac{dp'}{dq},$$

where dp/dq and dp'/dq in $L^1(q)$ are the Radon-Nikodym derivatives of p and p' , respectively. In this case, the symmetric part of \succsim_{cx} coincides with the identical distribution of the densities wrt q . For example, if S is finite and q the uniform, then $p \sim_{cx} p'$ if and only if there is a permutation $\pi : S \rightarrow S$ such that $p' = p \circ \pi$.

A function $T : \Delta \rightarrow (-\infty, \infty]$, with $\text{dom } T \subseteq \Delta^\sigma(q)$, is

(i) *rearrangement invariant* (wrt q) if $p \sim_{cx} p' \implies T(p) = T(p')$;

(ii) *Schur convex* (wrt q) if $p \succsim_{cx} p' \implies T(p) \geq T(p')$.

We are ready to state our main result. A final piece of notation: given $\varphi \in L^1(q)$, its inverse distribution function $F_\varphi^{-1} : [0, 1] \rightarrow [-\infty, \infty]$ is defined by $F_\varphi^{-1}(\omega) = \inf \{x \in \mathbb{R} : q(\{s \in S : \varphi(s) \leq x\}) \geq \omega\}$ for all $\omega \in [0, 1]$.

Theorem 2 *Let \succsim be a binary relation with uncertainty averse representation (u, G) . Then, the following conditions are equivalent (wrt q):*

(i) \succsim satisfies second order stochastic dominance;

(ii) $G(t, \cdot)$ is Schur convex on Δ for all $t \in \mathbb{R}$.

⁷In (11) and (12) we assume that \succsim restricted to X is represented by the affine utility function u .

⁸Clearly, $\Phi_{icv} \subseteq \Phi_{mi}$, and so second order stochastic dominance obviously implies the first one. Moreover, it can be shown that in conditions (11) and (12) is actually enough to consider strictly increasing functions.

In this case,

$$\min_{p \in \Delta} G \left(\int u(f) dp, p \right) = \min_{p \in \Delta^\sigma(q)} G \left(\int_0^1 F_{u \circ f}^{-1}(\omega) F_{\frac{dp}{dq}}^{-1}(1 - \omega) d\omega, p \right) \quad \forall f \in \mathcal{F} \quad (14)$$

and

$$G(t, p) = \begin{cases} \sup \left\{ u(x_f) : \int_0^1 F_{u \circ f}^{-1}(\omega) F_{\frac{dp}{dq}}^{-1}(1 - \omega) d\omega < t \right\} & \text{if } (t, p) \in \mathbb{R} \times \Delta^\sigma(q) \\ \infty & \text{else.} \end{cases} \quad (15)$$

Moreover, if (S, Σ, q) is adequate, then (i) and (ii) are equivalent to:

(iii) \succsim satisfies first order stochastic dominance;

(iv) \succsim is probabilistically sophisticated;

(v) $G(t, \cdot)$ is rearrangement invariant on Δ for all $t \in \mathbb{R}$.

Theorem 2 is a powerful result that establishes few nontrivial properties. To fix ideas, consider the important case where (S, Σ, q) is adequate. First, Theorem 2 shows that in this case probabilistic sophistication remarkably turns out to be equivalent to first and to second order stochastic dominance, something that in general does not hold. Second, Theorem 2 shows that Schur convexity and rearrangement invariance are the functional properties of the index G that characterize a probabilistically sophisticated and uncertainty averse preference. Finally, (14) and (15) show what form the index G takes for this class of preferences.

Taken together, all these features of Theorem 2 establish for the adequate case a complete characterization of uncertainty averse preferences that are probabilistically sophisticated. In particular, the equivalence between (iv) and (v), that is, between probabilistic sophistication and rearrangement invariance, extends to the present much more general setting the characterization of probabilistic sophistication via a symmetry property of the index c in (6) that [18] established for variational preferences.

Observe that the first part of the theorem fully characterizes second order stochastic dominance even in the nonadequate case. Since in the adequate case second order stochastic dominance is equivalent to probabilistic sophistication, in a sense Theorem 2 can be more properly viewed as a characterization of uncertainty averse preferences that satisfy second order stochastic dominance, a very important property in economic applications.

We close with a corollary of our main result. Denote by $\succsim_{u,q}$ the Subjective Expected Utility (SEU) preference represented by $\int u(f) dq$.

Corollary 1 *Let \succsim be a binary relation with uncertainty averse representation (u, G) . Then, \succsim is more uncertainty averse than $\succsim_{u,q}$ provided at least one of the following conditions holds (wrt q):*

(i) \succsim satisfies second order stochastic dominance;

(ii) \succsim is probabilistically sophisticated and (S, Σ, q) is adequate.

That is, according to the terminology of [8], each of the conditions (i) and (ii) guarantees that \succsim is an absolute uncertainty averse preference.

4 Unambiguous Events

Marinacci [17] pointed out a possible tension between probabilistic sophistication, which is based on a single reference probability, and the multiple priors representation, which instead relies on several possible probabilities. Thank to Theorem 2, in this section we show that this possible tension holds, much more generally, among probabilistic sophistication and uncertainty averse representations.

In order to do so, we first extend to our setting the notion of nontrivial unambiguous event of [17]. Consider a multiple priors representation a la Gilboa and Schmeidler [10]

$$V(f) = \min_{p \in C} \int u(f) dp \quad \forall f \in \mathcal{F}, \quad (16)$$

where C is a weak* closed set of Δ . An event is nontrivial and unambiguous if and only if $0 < p(A) = p'(A) < 1$ for all $p, p' \in C$. To generalize this notion to the present setting, consider the revealed unambiguous preference of Ghirardato, Maccheroni, and Marinacci [9], defined as

$$f \succsim^* g \iff \lambda f + (1 - \lambda) h \succ \lambda g + (1 - \lambda) h \quad \forall h \in \mathcal{F}, \forall \lambda \in (0, 1]. \quad (17)$$

In [2, Theorem 10] we show that for preferences with an uncertainty averse representation it holds

$$f \succsim^* g \iff \int u(f) dp \geq \int u(g) dp \quad \forall p \in \text{dom}_\Delta G, \quad (18)$$

where $\text{dom}_\Delta G = \{p \in \Delta : G(t, p) < \infty \text{ for some } t \in \mathbb{R}\}$. This motivates the following definition.

Definition 2 *Let \succsim be an uncertainty averse preference. An event A in Σ is nontrivial and unambiguous if there exist $x, y, z \in X$ such that $x \succ z \succ y$ and $xAy \sim^* z$.*

In other words, an event A is unambiguous if the act xAy is “unambiguously” indifferent to a constant act z . Clearly, constant acts are unambiguous since their outcomes are independent of the underlying state space realizations. Moreover, the condition $x \succ z \succ y$ rules out the possibility that A is “unambiguous” because either A or its complement are deemed null wrt \succsim .

Proposition 2 *Let \succsim be a binary relation with uncertainty averse representation (u, G) . Then, the following properties are equivalent:*

- (i) A is unambiguous;
- (ii) for each $x, y \in X$ such that $x \succ y$ there exists $z \in X$ such that $x \succ z \succ y$ and $xAy \sim^* z$;
- (iii) $0 < p(A) = p'(A) < 1$ for all $p, p' \in \text{dom}_\Delta G$.

Remark Strazlecki [23] implicitly provides different notions of unambiguous event for unbounded variational preferences. By (18) and Proposition 2-(iii), it follows that our notion of unambiguous event gives a behavioral foundation and a generalization of the notion contained in his Assumption 2N.

We now state the main result of this section.

Proposition 3 *Let \succsim be a binary relation with uncertainty averse representation (u, G) . If there exists a nontrivial unambiguous event, then (wrt q):*

- (i) \succsim satisfies second order stochastic dominance if and only if \succsim is the SEU preference $\succsim_{u,q}$.

(ii) \succsim is probabilistically sophisticated if and only if \succsim is the SEU preference $\succsim_{u,q}$, provided (S, Σ, q) is adequate.

Point (ii) generalizes the main result of [17] to the present general setting. Point (i) shows that, even when (S, Σ, q) is not adequate, second order stochastic dominance and uncertainty aversion can be both satisfied only by SEU preference as soon as there exists at least one nontrivial unambiguous event.

By Theorem 2, probabilistic sophistication and second order stochastic dominance are equivalent properties in the adequate case. Point (i) thus shows that the tension originally identified by [17] among probabilistic sophistication and multiple priors holds much more generally among second order stochastic dominance and uncertainty aversion. Since second order stochastic dominance is a widely used property in applications, this is an important novel insight of Proposition 3. Along with its substantially greater generality, this insight is what makes Proposition 3 a significant advance relative to the analysis of [17].

A Proofs and Related Analysis

A.1 Proof of Proposition 1

(i) implies (ii). By Savage's Expected Utility Theorem,⁹ there are a nonconstant $v : X \rightarrow \mathbb{R}$ and a nonatomic probability q on Σ such that $V : \mathcal{F} \rightarrow \mathbb{R}$ given by $V(f) = \int v(f) dq$ represents \succsim . In particular, \succsim satisfies A1 and A2, which together with A4 and A5, guarantee that:

- There exists a nonconstant affine function $u : X \rightarrow \mathbb{R}$ and a function $I : B_0(u(X)) \rightarrow \mathbb{R}$ normalized, monotone, and continuous such that $f \succsim g \iff I(u(f)) \geq I(u(g))$. Moreover, u is cardinally unique, and, given u , there is a unique normalized $I : B_0(u(X)) \rightarrow \mathbb{R}$ that represents \succsim in the above sense (see [2, Lemma 60]).
- For each $f \in \mathcal{F}$ there exists $x_f \in X$ such that $f \sim x_f$.

Since u is affine, $u(X) = K$ is an interval. Since both u and v represent \succsim on X , there exists a strictly increasing $\phi : u(X) \rightarrow \mathbb{R}$ such that $v = \phi \circ u$. It only remains to show that ϕ is continuous. For all $\psi \in B_0(K)$, let $f \in \mathcal{F}$ and x_f in X be such that $\psi = u(f)$ and $x_f \sim f$. Then

$$\int \phi(\psi) dq = \int \phi(u(f)) dq = V(f) = v(x_f) = \phi(u(x_f)),$$

and so $\int \phi(\psi) dq \in \text{Im } \phi$.¹⁰ Now, for each $t_1 = \phi(k_1), t_2 = \phi(k_2) \in \text{Im } \phi$ and $\alpha \in (0, 1)$, take $A \in \Sigma$ such that $q(A) = \alpha$ (this is possible since q is nonatomic). Then

$$\alpha t_1 + (1 - \alpha) t_2 = \alpha \phi(k_1) + (1 - \alpha) \phi(k_2) = \int \phi(k_1 1_A + k_2 1_{A^c}) dq \in \text{Im } \phi.$$

Therefore, $\text{Im } \phi$ is convex and ϕ is continuous (ϕ is increasing).

(ii) implies (i). Clearly, (8) is equivalent to $f \succsim g \iff \int \phi(u(f)) dq \geq \int \phi(u(g)) dq$. Thus, P1-P6 hold. Moreover, A4 follows from the fact that $V : X \rightarrow \mathbb{R}$ given by $V(x) = \phi(u(x))$ represents \succsim on X , with ϕ strictly increasing and u affine.

⁹Notice that we are assuming that Σ is a σ -algebra (see, e.g., Wakker [24, Observation 2]).

¹⁰In particular, $\phi^{-1}(\int \phi(\psi) dq) = u(x_f) = I(u(x_f)) = I(u(f)) = I(\psi)$.

It remains to show that $\psi \mapsto \int \phi(\psi) dq$ is continuous on $B_0(K)$, which in turn implies A5. Let ψ_n be a sequence in $B_0(K)$ that supnorm converges to $\psi \in B_0(K)$. For each $\delta > 0$, eventually

$$|\psi_n(s) - \psi(s)| \leq \delta \quad \forall s \in S. \quad (19)$$

Moreover, ψ_n is supnorm bounded and so it is easy to check that there are $a, b \in \mathbb{R}$ such that $[a, b] \subseteq K$ and, eventually, $\psi_n, \psi \in B_0([a, b])$ for all $n \geq 1$. But, being continuous, ϕ is also uniformly continuous on $[a, b]$. Thus, for all $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that

$$t, r \in [a, b] \text{ and } |t - r| \leq \delta_\varepsilon \implies |\phi(t) - \phi(r)| \leq \varepsilon$$

Then, eventually $|\psi_n(s) - \psi(s)| \leq \delta_\varepsilon$ for all $s \in S$, and $|\phi(\psi_n(s)) - \phi(\psi(s))| \leq \varepsilon$ for all $s \in S$. That is, $B_0(\phi(K)) \ni \phi(\psi_n) \rightarrow \phi(\psi)$ and $\int \phi(\psi_n) dq \rightarrow \int \phi(\psi) dq$, as wanted. Let $f, g, h \in \mathcal{F}$, and $\{\alpha_n\} \in [0, 1]$ be such that $\alpha_n f + (1 - \alpha_n)g \succsim h$ for all $n \geq 1$, and assume $\alpha_n \rightarrow \alpha$. Then

$$\int \phi(u(\alpha_n f + (1 - \alpha_n)g)) dq = V(\alpha_n f + (1 - \alpha_n)g) \geq V(h) \quad \forall n \geq 1. \quad (20)$$

But, $u(\alpha_n f + (1 - \alpha_n)g) = \alpha_n u(f) + (1 - \alpha_n)u(g) = u(g) + \alpha_n(u(f) - u(g)) \rightarrow u(\alpha f + (1 - \alpha)g)$ in the supnorm. Thus, passing to the limits in (20),

$$V(\alpha f + (1 - \alpha)g) = \int \phi(u(\alpha f + (1 - \alpha)g)) dq \geq V(h),$$

which immediately delivers A5.

As to uniqueness, we show that $(\bar{q}, \bar{u}, \bar{\phi})$ represents \succsim in the sense of (8) if and only if $\bar{q} = q$ and there exist $\alpha, \beta, \eta, \kappa \in \mathbb{R}$ with $\alpha, \eta > 0$ such that for all $x \in X$ and $t \in \bar{u}(X)$:

$$\bar{u}(x) = \frac{u(x) - \kappa}{\eta} \quad \text{and} \quad \bar{\phi}(t) = \alpha(\phi(\eta t + \kappa)) + \beta. \quad (21)$$

By Savage's Expected Utility Theorem, $\bar{q} = q$ and there are $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $\bar{\phi} \circ \bar{u} = \alpha(\phi \circ u) + \beta$. By the von Neumann-Morgenstern's Expected Utility Theorem, there are $\eta > 0$ and $\kappa \in \mathbb{R}$ such that $\bar{u} = \eta^{-1}(u - \kappa)$. Therefore, $\bar{\phi}(\bar{u}(x)) = \alpha(\phi(u(x))) + \beta = \alpha(\phi(\eta \bar{u}(x) + \kappa)) + \beta$ for all $x \in X$, and $\bar{\phi}(t) = \alpha(\phi(\eta t + \kappa)) + \beta$ for all $t \in \bar{u}(X)$. The converse is easily checked.

Next we show that A3 implies concavity of ϕ and A7 implies $q \in \Delta^\sigma$, the converse implications being trivial. Assume per contra that A3 holds and that ϕ is not concave. Since ϕ is continuous, there are $r, t \in K$ such that $\phi(2^{-1}t + 2^{-1}r) < 2^{-1}\phi(t) + 2^{-1}\phi(r)$. Let $H \in \Sigma$ be such that $q(H) = 2^{-1}$, and $x, y \in X$ be such that $u(x) = r$ and $u(y) = t$. Then

$$\begin{aligned} V(xHy) &= \int \phi(u(xHy)) dq = \int \phi(u(x)) 1_H + \phi(u(y)) 1_{H^c} dq = \frac{1}{2}\phi(r) + \frac{1}{2}\phi(t) \\ &= \frac{1}{2}\phi(t) + \frac{1}{2}\phi(r) = V(yHx) \end{aligned}$$

and we get the following violation of A3:

$$V\left(\frac{1}{2}xHy + \frac{1}{2}yHx\right) = V\left(\frac{1}{2}x + \frac{1}{2}y\right) = \phi\left(u\left(\frac{1}{2}x + \frac{1}{2}y\right)\right) = \phi\left(\frac{r}{2} + \frac{t}{2}\right) < V(xHy).$$

Suppose A7 holds and let $\Sigma \ni E_n \searrow \emptyset$. Choose $z \succ y$ and consider the sequence $z_m = (1 - m^{-1})z + m^{-1}y$ for all $m \geq 1$. We have $u(z_m) = u(z) - \frac{1}{m}(u(z) - u(y)) < u(z)$. For all $m \geq 1$, $z \succ z_m$ and there is $n_m \geq 1$ such that $yE_{n_m}z \succ z_m$, i.e.,

$$q(E_{n_m})\phi(u(y)) + (1 - q(E_{n_m}))\phi(u(z)) > \phi(u(z_m)). \quad (22)$$

Wlog set $\phi(u(y)) = 0 = 1 - \phi(u(z))$. Thus, $w_n = \phi(u(z_m)) \rightarrow 1$. By (22), for all $m \geq 1$ there is $n_m \geq 1$ such that $1 - q(E_{n_m}) > w_m$, i.e., $q(E_{n_m}) < 1 - w_m$. But, $q(E_k)$ is a decreasing sequence, therefore $0 \leq \lim_k q(E_k) \leq q(E_{n_m}) < 1 - w_m$ for all $m \geq 1$. Thus, $\lim_k q(E_k) = 0$ and $q \in \Delta^\sigma$. \blacksquare

A.2 Proof of Theorem 2

In this appendix we prove the main result of Section 3. Let $(u, G) \in \mathcal{U}(X) \times \mathcal{L}(\mathbb{R} \times \Delta)$ be an uncertainty averse representation of a preference \succsim in the sense of Definition 1 and set

$$I(\varphi) = \min_{p \in \Delta} G\left(\int \varphi dp, p\right) \quad \forall \varphi \in B_0. \quad (23)$$

By [2, Theorem 53] there exists at least one $q \in \Delta^\sigma$ such that $\text{dom } G \subseteq \mathbb{R} \times \Delta^\sigma(q)$, and hence

$$I(\varphi) = \min_{p \in \Delta^\sigma(q)} G\left(\int \varphi dp, p\right) \quad \forall \varphi \in B_0.$$

Notice that, by Theorem 1,

$$G(t, p) = \sup \left\{ I(\varphi) : \int \varphi dp \leq t \right\} \quad \forall (t, p) \in \mathbb{R} \times \Delta.$$

In the study of rearrangement invariance it is useful to consider some important stochastic orders. We already introduced in (13) the convex order \succsim_{cx} on $L^1(q)$. The *increasing convex order* \succsim_{icx} , the *first order stochastic dominance* (*fsd*), and the *second order stochastic dominance* (*ssd*) are defined analogously by replacing the set of convex functions Φ_{cx} with that of increasing convex functions Φ_{icx} , increasing functions Φ_{mi} , and increasing concave functions Φ_{icv} . Notice that $\varphi \succsim_{icx} \psi$ if and only if $-\varphi \succsim_{ssd} -\psi$, and that the preorders \succsim_{cx} , \succsim_{icx} , \succsim_{fsd} , and \succsim_{ssd} all share the same symmetric part \sim_d , which is the *identical distribution* relation wrt q .¹¹

A function J defined on a subset of $L^1(q)$ with values in $(-\infty, \infty]$ is:

1. *rearrangement invariant* if $\varphi \sim_d \psi \implies J(\varphi) = J(\psi)$;
2. *Schur convex* if $\varphi \succsim_{cx} \psi \implies J(\varphi) \geq J(\psi)$

Moreover, J *preserves first* (resp., *second*) *order stochastic dominance* if $\varphi \succsim_{fsd} \psi$ (resp., $\varphi \succsim_{ssd} \psi$) implies $J(\varphi) \geq J(\psi)$.

Theorem 3 *Let I be the function defined by (23) and $q \in \Delta^\sigma$ be such that $\text{dom } G \subseteq \mathbb{R} \times \Delta^\sigma(q)$. The following conditions are equivalent (wrt q):*

- (i) I *preserves second order stochastic dominance on B_0 ;*
- (ii) $G(t, \cdot)$ *is Schur convex on Δ for all $t \in \mathbb{R}$.*

In this case,

$$I(\varphi) = \min_{p \in \Delta^\sigma(q)} G\left(\int_0^1 F_\varphi^{-1}(\omega) F_{\frac{dp}{dq}}^{-1}(1 - \omega) d\omega, p\right) \quad \forall \varphi \in B_0 \quad (24)$$

and

$$G(t, p) = \begin{cases} \sup \left\{ I(\psi) : \int_0^1 F_\psi^{-1}(\omega) F_{\frac{dp}{dq}}^{-1}(1 - \omega) d\omega < t \right\} & \text{if } (t, p) \in \mathbb{R} \times \Delta^\sigma(q) \\ \infty & \text{else.} \end{cases} \quad (25)$$

Moreover, if (S, Σ, q) is adequate, then (i) and (ii) are equivalent to:

- (iii) I *preserves first order stochastic dominance on B_0 ;*

¹¹See Chong (1974) for this fact and for alternative characterizations of some of these stochastic orders.

(iv) I is rearrangement invariant on B_0 ;

(v) $G(t, \cdot)$ is rearrangement invariant on Δ for all $t \in \mathbb{R}$.

For all $\varphi \in L^1(q)$ and all $\omega \in [0, 1]$, set

$$\delta_\varphi(\omega) = \inf \{x \in \mathbb{R} : q(\{s \in S : \varphi(s) > x\}) \leq \omega\} \quad (= \inf \{x \in \mathbb{R} : F_\varphi(x) \geq 1 - \omega\} = F_\varphi^{-1}(1 - \omega)).$$

Proof. The proof relies on the theory of rearrangement invariant Banach spaces developed by Luxemburg [14] and Chong and Rice [7].

Step 1. If $\psi \in B_0$ and $p \in \Delta^\sigma(q)$, then

$$\left\{ \int \psi dp' : \Delta^\sigma(q) \ni p' \preceq_{cx} p \right\} = \left[\int_0^1 \delta_\psi(\omega) \delta_{\frac{dp}{dq}}(1 - \omega) d\omega, \int_0^1 \delta_\psi(\omega) \delta_{\frac{dp}{dq}}(\omega) d\omega \right]. \quad (26)$$

Moreover, if (S, Σ, q) is adequate, then

$$\int_0^1 \delta_\psi(\omega) \delta_{\frac{dp}{dq}}(1 - \omega) d\omega = \min \left\{ \int \psi dp' : \Delta^\sigma(q) \ni p' \sim_d p \right\} \text{ and} \quad (27)$$

$$\int_0^1 \delta_\psi(\omega) \delta_{\frac{dp}{dq}}(\omega) d\omega = \max \left\{ \int \psi dp' : \Delta^\sigma(q) \ni p' \sim_d p \right\}. \quad (28)$$

Proof. [7, 10.2, 13.4, and 13.8] guarantee that, if $\varphi, \psi \in L^1(q)$ and $\delta_{|\psi|}\delta_{|\varphi|} \in L^1([0, 1], \mathcal{B}, \lambda) = L^1(\lambda)$, then

$$\left\{ \int \psi \varphi' dq : L^1(q) \ni \varphi' \preceq_{cx} \varphi \right\} = \left[\int_0^1 \delta_\psi(\omega) \delta_\varphi(1 - \omega) d\omega, \int_0^1 \delta_\psi(\omega) \delta_\varphi(\omega) d\omega \right]. \quad (29)$$

Moreover, if (S, Σ, q) is adequate, then

$$\int_0^1 \delta_\psi(\omega) \delta_\varphi(1 - \omega) d\omega = \min \left\{ \int \psi \varphi' dq : L^1(q) \ni \varphi' \sim_d \varphi \right\} \text{ and} \quad (30)$$

$$\int_0^1 \delta_\psi(\omega) \delta_\varphi(\omega) d\omega = \max \left\{ \int \psi \varphi' dq : L^1(q) \ni \varphi' \sim_d \varphi \right\}. \quad (31)$$

Notice that, the condition $\delta_{|\psi|}\delta_{|\varphi|} \in L^1(\lambda)$ is implied by $\delta_{|\psi|} \in L^\infty(\lambda)$ and $\delta_{|\varphi|} \in L^1(\lambda)$, which is implied by $\psi \in B_0$ and $\varphi \in L^1(q)$ [7, 4.3].

If, in addition, φ is a probability density (p.d.) and $\varphi' \preceq_{cx} \varphi$, then $\text{essinf } \varphi' \geq 0$ [7, 10.2] and $\int \varphi' dq = \int \varphi dq = 1$, i.e., φ' is a probability density.

Finally, if $\psi \in B_0$ and $p \in \Delta^\sigma(q)$, then

$$\begin{aligned} \left\{ \int \psi dp' : \Delta^\sigma(q) \ni p' \preceq_{cx} p \right\} &= \left\{ \int \psi \varphi' dq : \varphi' \text{ is a p.d. and } \varphi' \preceq_{cx} \frac{dp}{dq} \right\} \\ &= \left\{ \int \psi \varphi' dq : L^1(q) \ni \varphi' \preceq_{cx} \frac{dp}{dq} \right\} \\ &= \left[\int_0^1 \delta_\psi(\omega) \delta_{\frac{dp}{dq}}(1 - \omega) d\omega, \int_0^1 \delta_\psi(\omega) \delta_{\frac{dp}{dq}}(\omega) d\omega \right]. \end{aligned}$$

Moreover, if (S, Σ, q) is adequate, then

$$\begin{aligned} \int_0^1 \delta_\psi(\omega) \delta_{\frac{dp}{dq}}(1 - \omega) d\omega &= \min \left\{ \int \psi \varphi' dq : L^1(q) \ni \varphi' \sim_d \frac{dp}{dq} \right\} \\ &= \min \left\{ \int \psi \varphi' dq : \varphi' \text{ is a p.d. and } \varphi' \sim_d \frac{dp}{dq} \right\} \\ &= \min \left\{ \int \psi dp' : \Delta^\sigma(q) \ni p' \sim_d p \right\} \end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \delta_\psi(\omega) \delta_{\frac{dp}{dq}}(\omega) d\omega &= \max \left\{ \int \psi \varphi' dq : L^1(q) \ni \varphi' \sim_d \frac{dp}{dq} \right\} \\
&= \max \left\{ \int \psi \varphi' dq : \varphi' \text{ is a p.d. and } \varphi' \sim_d \frac{dp}{dq} \right\} \\
&= \max \left\{ \int \psi dp' : \Delta^\sigma(q) \ni p' \sim_d p \right\}
\end{aligned}$$

as wanted. \square

The next step is essentially due to Hardy.

Step 2. Let $r = \infty$ and $\bar{r} = 1$ or viceversa, $\varphi, \varphi' \in L^r(q)$ and $\psi \in L^{\bar{r}}(q)$.

- (a) $\varphi \preceq_{cx} \varphi'$ implies $\int_0^1 \delta_\varphi(\omega) \delta_\psi(\omega) d\omega \leq \int_0^1 \delta_{\varphi'}(\omega) \delta_\psi(\omega) d\omega$.
- (b) $\varphi \preceq_{cx} \varphi'$ implies $\int_0^1 \delta_\varphi(\omega) \delta_\psi(1-\omega) d\omega \geq \int_0^1 \delta_{\varphi'}(\omega) \delta_\psi(1-\omega) d\omega$.
- (c) $\varphi \preceq_{icx} \varphi'$ and $\psi \geq 0$ (q -a.e.) implies $\int_0^1 \delta_\varphi(\omega) \delta_\psi(\omega) d\omega \leq \int_0^1 \delta_{\varphi'}(\omega) \delta_\psi(\omega) d\omega$.

Proof. See [7, 9.1] \square

Step 3. If either $G(t, \cdot)$ is Shur convex on Δ for all $t \in \mathbb{R}$, or (S, Σ, q) is adequate and $G(t, \cdot)$ is rearrangement invariant on Δ for all $t \in \mathbb{R}$, then

$$I(\varphi) = \min_{p \in \Delta^\sigma(q)} G\left(\int_0^1 \delta_\varphi(\omega) \delta_{\frac{dp}{dq}}(1-\omega) d\omega, p\right) \quad \forall \varphi \in B_0. \quad (32)$$

Proof. Let $\varphi \in B_0$. Then, by (26), $\int \varphi dp \geq \int_0^1 \delta_\varphi(\omega) \delta_{\frac{dp}{dq}}(1-\omega) d\omega$ for all $p \in \Delta^\sigma(q)$. Thus, monotonicity of G in the first component implies

$$I(\varphi) = \min_{p \in \Delta^\sigma(q)} G\left(\int \varphi dp, p\right) \geq \inf_{p \in \Delta^\sigma(q)} G\left(\int_0^1 \delta_\varphi(\omega) \delta_{\frac{dp}{dq}}(1-\omega) d\omega, p\right).$$

Conversely, by (26), for any $p \in \Delta^\sigma(q)$ there exists $p' \preceq_{cx} p$ (resp., by (27) there exists $p' \sim_d p$) such that

$$\int_0^1 \delta_\varphi(\omega) \delta_{\frac{dp}{dq}}(1-\omega) d\omega = \int \varphi dp'.$$

Thus,

$$G\left(\int_0^1 \delta_\varphi(\omega) \delta_{\frac{dp}{dq}}(1-\omega) d\omega, p\right) = G\left(\int \varphi dp', p\right) \geq G\left(\int \varphi dp', p'\right) \geq I(\varphi)$$

by Shur convexity (resp., rearrangement invariance).

Therefore,

$$\inf_{p \in \Delta^\sigma(q)} G\left(\int_0^1 \delta_\varphi(\omega) \delta_{\frac{dp}{dq}}(1-\omega) d\omega, p\right) \geq I(\varphi)$$

and the infimum is attained. \square

Step 4. (ii) implies (i) and (24), also (v) implies (i) and (24) provided (S, Σ, q) is adequate.

Proof. By Step 3, (ii) guarantees that (32) holds and the same is true for (v) if (S, Σ, q) is adequate.

But, for all $\varphi \in B_0$ and $p \in \Delta^\sigma(q)$,

$$\int_0^1 \delta_\varphi(\omega) \delta_{\frac{dp}{dq}}(1-\omega) d\omega = \int_0^1 \delta_\varphi(1-\omega) \delta_{\frac{dp}{dq}}(\omega) d\omega = \int_0^1 F_\varphi^{-1}(\omega) F_{\frac{dp}{dq}}^{-1}(1-\omega) d\omega$$

which plugged in (32) delivers (24).

Moreover, $\varphi \succsim_{ssd} \psi$ if and only if $-\varphi \precsim_{icx} -\psi$. Thus, Step 2.c implies $\int_0^1 \delta_{-\varphi}(\omega) \delta_{dp/dq}(\omega) d\omega \leq \int_0^1 \delta_{-\psi}(\omega) \delta_{dp/dq}(\omega) d\omega$ for all $p \in \Delta^\sigma(q)$, but $\delta_{-\varphi}(\omega) = -\delta_\varphi(1-\omega)$ (λ -a.e.) [7, 4.4] and the same is true for ψ . This implies that $\int_0^1 -\delta_\varphi(1-\omega) \delta_{dp/dq}(\omega) d\omega \leq \int_0^1 -\delta_\psi(1-\omega) \delta_{dp/dq}(\omega) d\omega$ and hence $\int_0^1 \delta_\varphi(\omega) \delta_{dp/dq}(1-\omega) d\omega \geq \int_0^1 \delta_\psi(\omega) \delta_{dp/dq}(1-\omega) d\omega$ for all $p \in \Delta^\sigma(q)$. By (32), monotonicity of G allows to conclude that

$$I(\varphi) = \min_{p \in \Delta^\sigma(q)} G\left(\int_0^1 \delta_\varphi(\omega) \delta_{\frac{dp}{dq}}(1-\omega) d\omega, p\right) \geq \min_{p \in \Delta^\sigma(q)} G\left(\int_0^1 \delta_\psi(\omega) \delta_{\frac{dp}{dq}}(1-\omega) d\omega, p\right) = I(\psi).$$

Therefore, I preserves second order stochastic dominance and, in particular, it is rearrangement invariant. \square

Step 5. If $\varphi \in L^\infty(q)$ then there exists $\{\varphi_n\} \subseteq B_0$ such that φ_n is the conditional expectation of φ on a finite σ -algebra for all $n \in \mathbb{N}$ and $\varphi_n \xrightarrow{\|\cdot\|_\infty} \varphi$. In particular, $\varphi \succsim_{cx} \varphi_n$ for all $n \in \mathbb{N}$.

Proof. Let $\varphi \in L^\infty(q)$ and wlog take a bounded version of φ . There exists $\{\psi_n\} \subseteq B_0$ that uniformly converges to φ . Set, for each $n \in \mathbb{N}$, $d_n = \|\varphi - \psi_n\|$, $\psi_n^o = \psi_n - d_n$, $\psi_n' = \psi_n + d_n$, $\Sigma_n = \sigma(\psi_n) = \sigma(\psi_n^o) = \sigma(\psi_n')$. It is immediate to see that $\psi_n^o \leq \varphi \leq \psi_n'$ for all $n \in \mathbb{N}$. Moreover, both $\{\psi_n^o\}$ and $\{\psi_n'\}$ converge uniformly to φ , and, for each $n \in \mathbb{N}$, there exist suitable versions of $\mathbb{E}(\psi_n^o|\Sigma_n)$, $\mathbb{E}(\varphi|\Sigma_n)$, and $\mathbb{E}(\psi_n'|\Sigma_n)$ such that $\psi_n = \mathbb{E}(\psi_n^o|\Sigma_n) \leq \mathbb{E}(\varphi|\Sigma_n) \leq \mathbb{E}(\psi_n'|\Sigma_n) = \psi_n'$. Define $\varphi_n = \mathbb{E}(\varphi|\Sigma_n)$ for all $n \in \mathbb{N}$. Clearly, $\varphi_n \in B_0$ and it uniformly converges to φ .

Finally, observe that for all convex functions $\ell : \mathbb{R} \rightarrow \mathbb{R}$, by Jensen's inequality, we have that q -a.e.

$$\ell(\varphi_n) = \ell(\mathbb{E}(\varphi|\Sigma_n)) \leq \mathbb{E}(\ell(\varphi)|\Sigma_n) \quad \forall n \in \mathbb{N}.$$

Then, by integrating both sides, for all convex $\ell : \mathbb{R} \rightarrow \mathbb{R}$

$$\mathbb{E}(\ell(\varphi_n)) \leq \mathbb{E}(\mathbb{E}(\ell(\varphi)|\Sigma_n)) = \mathbb{E}(\ell(\varphi)) \quad \forall n \in \mathbb{N}.$$

\square

Step 6. Let $\psi \in B_0$ and $p \in \Delta^\sigma(q)$. Then,

$$\text{cl}_{L^\infty(q)}(\{\varphi \in B_0 : \varphi \precsim_{cx} \psi\}) = \{\varphi \in L^\infty(q) : \varphi \precsim_{cx} \psi\}. \quad (33)$$

In particular,

$$\inf \left\{ \int \varphi dp : B_0 \ni \varphi \precsim_{cx} \psi \right\} = \min \left\{ \int \varphi dp : L^\infty(q) \ni \varphi \precsim_{cx} \psi \right\} = \int_0^1 \delta_{\frac{dp}{dq}}(\omega) \delta_\psi(1-\omega) d\omega. \quad (34)$$

Moreover, if (S, Σ, q) is adequate then

$$\min \left\{ \int \varphi dp : B_0 \ni \varphi \sim_d \psi \right\} = \min \left\{ \int \varphi dp : L^\infty(q) \ni \varphi \sim_d \psi \right\} = \int_0^1 \delta_{\frac{dp}{dq}}(\omega) \delta_\psi(1-\omega) d\omega. \quad (35)$$

Proof. It is easy to verify that $\{\varphi \in L^\infty(q) : \varphi \precsim_{cx} \psi\}$ is closed wrt $\|\cdot\|_\infty$. Therefore,

$$\text{cl}_{L^\infty(q)}(\{\varphi \in B_0 : \varphi \precsim_{cx} \psi\}) \subseteq \{\varphi \in L^\infty(q) : \varphi \precsim_{cx} \psi\}.$$

Conversely, by Step 5, for all $\varphi \in L^\infty(q)$ such that $\varphi \precsim_{cx} \psi$ there exists $\{\varphi_n\} \subseteq B_0$ such that $\varphi_n \xrightarrow{\|\cdot\|_\infty} \varphi$ and $\varphi_n \precsim_{cx} \varphi \precsim_{cx} \psi$ for all $n \in \mathbb{N}$. Hence, (33) follows.

Moreover, [7, 4.3, 10.2, and 13.8] guarantee that, if $\psi \in B_0$ and $p \in \Delta^\sigma(q)$ then $\delta_{|\psi|} \delta_{\left|\frac{dp}{dq}\right|} \in L^1(\lambda)$ and

$$\left\{ \int \varphi \frac{dp}{dq} dq : L^\infty(q) \ni \varphi \preceq_{cx} \psi \right\} = \left[\int_0^1 \delta_{\frac{dp}{dq}}(\omega) \delta_\psi(1-\omega) d\omega, \int_0^1 \delta_{\frac{dp}{dq}}(\omega) \delta_\psi(\omega) d\omega \right]. \quad (36)$$

By (33) and (36), (34) follows since $\int \cdot dp$ is a continuous linear functional on $L^\infty(q)$.

If (S, Σ, q) is adequate, [7, 4.3, 10.2, and 13.4] guarantee that, if $\psi \in B_0$ and $p \in \Delta^\sigma(q)$ then $\delta_{|\psi|} \delta_{\left|\frac{dp}{dq}\right|} \in L^1(\lambda)$ and

$$\min \left\{ \int \varphi \frac{dp}{dq} dq : L^\infty(q) \ni \varphi \sim_d \psi \right\} = \int_0^1 \delta_{\frac{dp}{dq}} \delta_\psi(1-\omega)(\omega) d\omega.$$

But, notice that if ψ is simple and $L^\infty(q) \ni \varphi \sim_d \psi$, then there exists a version of φ which is simple too, thus proving (35). \square

Step 7. If either I preserves second order stochastic dominance or if (S, Σ, q) is adequate and I is rearrangement invariant, then, for all $(t, p) \in \mathbb{R} \times \Delta^\sigma(q)$,

$$G(t, p) = \sup \left\{ I(\psi) : \int \psi dp < t \right\} = \sup \left\{ I(\psi) : \int_0^1 \delta_{\frac{dp}{dq}}(\omega) \delta_\psi(1-\omega) d\omega < t \right\}. \quad (37)$$

Proof. Observe that the set $\{\psi \in B_0 : \int \psi dp \leq t\}$ is the closure of $\{\psi \in B_0 : \int \psi dp < t\}$. The fact that I is continuous implies the first equality. In what follows $\varphi, \psi \in B_0$. If I preserves second order stochastic dominance, then I is Schur concave. For, if $\varphi \preceq_{cx} \psi$ then $-\varphi \preceq_{cx} -\psi$ and $-\varphi \preceq_{icx} -\psi$. Hence, $\psi \preceq_{ssd} \varphi$. It follows that $I(\psi) \geq I(\varphi)$. Let I be Schur concave (resp., rearrangement invariant). Then,

$$\begin{aligned} \sup \left\{ I(\varphi) : \int \varphi dp < t \right\} &= \sup \left\{ I(\psi) : \text{there exists } \varphi \preceq_{cx} \psi \text{ s.t. } \int \varphi dp < t \right\} \\ (\text{resp., } &= \sup \left\{ I(\psi) : \text{there exists } \varphi \sim_d \psi \text{ s.t. } \int \varphi dp < t \right\}). \end{aligned}$$

But,

$$\begin{aligned} \sup \left\{ I(\psi) : \int \varphi dp < t \text{ for some } \varphi \preceq_{cx} \psi \right\} &= \sup \left\{ I(\psi) : \inf \left\{ \int \varphi dp : B_0 \ni \varphi \preceq_{cx} \psi \right\} < t \right\} \\ (\text{resp., } \sup \left\{ I(\psi) : \int \varphi dp < t \text{ for some } \varphi \sim_d \psi \right\} &= \sup \left\{ I(\psi) : \inf \left\{ \int \varphi dp : B_0 \ni \varphi \sim_d \psi \right\} < t \right\}). \end{aligned}$$

By Step 6, (37) follows. \square

Step 8. (i) implies (ii) and (25), also (iv) implies (ii) and (25) provided (S, Σ, q) is adequate.

Proof. By Step 7, (i) guarantees that (37) holds, and the same is true for (iii) if (S, Σ, q) is adequate. But, $\int_0^1 \delta_{\frac{dp}{dq}}(\omega) \delta_\psi(1-\omega) d\omega = \int_0^1 F_\psi^{-1}(\omega) F_{\frac{dp}{dq}}^{-1}(1-\omega) d\omega$ for all $\psi \in B_0$ and for all $p \in \Delta^\sigma(q)$. Hence, (37) implies (25). By Step 2.b and (37), it descends the following chain of implications

$$\begin{aligned} p \preceq_{cx} p' &\implies \int_0^1 \delta_{\frac{dp}{dq}}(\omega) \delta_\psi(1-\omega) d\omega \geq \int_0^1 \delta_{\frac{dp'}{dq}}(\omega) \delta_\psi(1-\omega) d\omega \text{ for all } \psi \in B_0 \\ &\implies \left\{ \psi \in B_0 : \int_0^1 \delta_{\frac{dp}{dq}}(\omega) \delta_\psi(1-\omega) d\omega < t \right\} \subseteq \left\{ \psi \in B_0 : \int_0^1 \delta_{\frac{dp'}{dq}}(\omega) \delta_\psi(1-\omega) d\omega < t \right\} \quad \forall t \in \mathbb{R} \\ &\implies G(t, p) \leq G(t, p') \quad \forall t \in \mathbb{R}. \end{aligned}$$

Hence, $G(t, \cdot)$ is Shur convex for all $t \in \mathbb{R}$. \square

Step 9. (i) implies (iii), (iii) implies (iv), and (ii) implies (v).

Proof. The step is proved by a routine argument. \square

Finally, Steps 4 and 8 guarantee that (i) \Leftrightarrow (ii). In this case (24) and (25) hold. Moreover, if (S, Σ, q) is adequate, the same steps and Step 9 deliver both (v) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (ii) \Rightarrow (v). \blacksquare

Before proving Theorem 2, we prove two ancillary results.

Lemma 1 *Let (u, G) be an uncertainty averse representation for \succsim . If \succsim is probabilistically sophisticated wrt $q \in \Delta^\sigma$, then $\text{dom } G \subseteq \mathbb{R} \times \Delta^\sigma(q)$.*

Proof. By definition, $\text{dom } G \subseteq \mathbb{R} \times \Delta^\sigma$. Next, we show that $G(t, p) = \infty$ for all $(t, p) \in \mathbb{R} \times \Delta^\sigma \setminus \Delta^\sigma(q)$. Fix $t \in \mathbb{R}$ and $p \in \Delta^\sigma \setminus \Delta^\sigma(q)$. It follows that there exists $A \in \Sigma$ such that $p(A) > 0$ and $q(A) = 0$. Since $u(X) = \mathbb{R}$, there exist $\{x_n\}, \{y_n\} \subseteq X$ such that $u(x_n) = \sqrt{n}$ and $u(y_n) = -n$. Define $f_n = y_n A x_n$ for all $n \in \mathbb{N}$. By probabilistic sophistication, it follows that $f_n \sim x_n$ for all $n \in \mathbb{N}$. Hence, $u(x_{f_n}) = u(x_n) = \sqrt{n}$ for all $n \in \mathbb{N}$. But, for each $n \in \mathbb{N}$

$$\int u(f_n) dp = -np(A) + \sqrt{n}p(A^c) \rightarrow -\infty \text{ as } n \rightarrow \infty.$$

It follows that eventually $f_n \in \{f \in \mathcal{F} : \int u(f) dp \leq t\}$ and $\sqrt{n} \in \{u(x_f) : \int u(f) dp \leq t\}$. We conclude that $G(t, p) = \infty$. \blacksquare

Lemma 2 *Let (u, G) be an uncertainty averse representation for \succsim and let I be defined as in (23). The following statements are true wrt $q \in \Delta^\sigma$.*

- (a) \succsim is probabilistically sophisticated if and only if I is rearrangement invariant on B_0 ;
- (b) \succsim satisfies first order stochastic dominance if and only if I preserves first order stochastic dominance on B_0 ;
- (c) \succsim satisfies second order stochastic dominance if and only if I preserves second order stochastic dominance on B_0 .

Proof. First notice that $B_0 = \{u(f) : f \in \mathcal{F}\}$. By definition,

$$V(f) = I(u(f)) \quad \forall f \in \mathcal{F}$$

represents \succsim .

(a) “Only if.” Consider $\varphi, \psi \in B_0$ such that $\varphi \sim_d \psi$, then

$$q(\{s \in S : \varphi(s) = t\}) = q(\{s \in S : \psi(s) = t\}) \quad \forall t \in \mathbb{R}.$$

Since $u(X) = \mathbb{R}$, for each $t \in \mathbb{R}$ choose $x_t \in X$ such that $u(x_t) = t$. Since φ and ψ are simple, it follows that $\varphi(S) = \{t_1, \dots, t_n\}$ and $\psi(S) = \{t'_1, \dots, t'_{n'}\}$. Define

$$A_i = \{s \in S : \varphi(s) = t_i\} \text{ and } B_j = \{s \in S : \psi(s) = t'_j\}.$$

Finally, define f and g such that $f(s) = x_{t_i}$ if $s \in A_i$ and $g(s) = x_{t'_j}$ if $s \in B_j$. It follows that $\varphi = u(f)$, $\psi = u(g)$ and f and g satisfy (9).¹² Thus, $f \sim g$. Therefore, $I(\varphi) = I(u(f)) = I(u(g)) = I(\psi)$.

¹²Notice that $q(f^{-1}(x_t)) = q(\varphi^{-1}(t)) = q(\psi^{-1}(t)) = q(g^{-1}(x_t))$ for all $t \in \mathbb{R}$, while $q(f^{-1}(x)) = 0 = q(g^{-1}(x))$ if $x \notin \{x_t\}_{t \in \mathbb{R}}$.

“If.” Suppose that f and g satisfy (9). Define $\varphi = u \circ f$ and $\psi = u \circ g$. Since φ and ψ are simple, it is immediate to see that $\varphi \sim_d \psi$. Then, since I is rearrangement invariant, it follows that $I(u(f)) = I(\varphi) = I(\psi) = I(u(g))$, which implies that $f \sim g$.

(b) “Only if.” Consider $\varphi, \psi \in B_0$ such that $q(\{s \in S : \varphi(s) \leq t\}) \leq q(\{s \in S : \psi(s) \leq t\})$ for each $t \in \mathbb{R}$. Define $f, g \in \mathcal{F}$ to be such that $\varphi = u(f)$ and $\psi = u(g)$. It follows that

$$\begin{aligned} q(\{s \in S : f(s) \preceq x\}) &= q(\{s \in S : u(f(s)) \leq u(x)\}) = q(\{s \in S : \varphi(s) \leq u(x)\}) \\ &\leq q(\{s \in S : \psi(s) \leq u(x)\}) = q(\{s \in S : u(g(s)) \leq u(x)\}) \\ &= q(\{s \in S : g(s) \preceq x\}) \quad \forall x \in X. \end{aligned}$$

Therefore, it is clear that f and g satisfy (10). It follows that $f \succeq g$, and so $I(\varphi) = I(u(f)) \geq I(u(g)) = I(\psi)$.

“If.” Suppose that f and g satisfy (10). Define $\varphi = u \circ f$ and $\psi = u \circ g$. It follows that

$$\begin{aligned} q(\{s \in S : \varphi(s) \leq t\}) &= q(\{s \in S : u(f(s)) \leq u(x_t)\}) = q(\{s \in S : f(s) \preceq x_t\}) \\ &\leq q(\{s \in S : g(s) \preceq x_t\}) = q(\{s \in S : u(g(s)) \leq u(x_t)\}) \\ &= q(\{s \in S : \psi(s) \leq t\}) \quad \forall t \in \mathbb{R}. \end{aligned}$$

It follows that $\varphi \succeq_{fsd} \psi$. Then, since I preserves first order stochastic dominance, it follows that $I(u(f)) = I(\varphi) \geq I(\psi) = I(u(g))$, which implies $f \succeq g$.

The proof of (c) is analogous. ■

Proof of Theorem 2. Consider the uncertainty averse representation (u, G) for \succeq . Then, define the functional $I : B_0 \rightarrow \mathbb{R}$ as in (23).

(i) implies (ii). Since \succeq satisfies second order stochastic dominance (wrt q), it is probabilistically sophisticated. By Lemma 1, $\text{dom } G \subseteq \mathbb{R} \times \Delta^\sigma(q)$. By Lemma 2, I preserves second order stochastic dominance. Hence, Theorem 3 guarantees that (ii) holds.

(ii) implies (i). Since $\text{dom } G \subseteq \mathbb{R} \times \Delta^\sigma(q)$, by Theorem 3 I preserves second order stochastic dominance. By Lemma 2, \succeq satisfies second order stochastic dominance.

Furthermore, assume (i) or (ii) hold. By Theorem 3, I satisfies (24) and (25). Hence, (14) and (15) follow from the observation that $I(u(f)) = \min_{p \in \Delta} G(\int u(f) dp, p) = u(x_f)$ for each $f \in \mathcal{F}$.

Assume that (S, Σ, q) is adequate.

(i) implies (iii) and (iii) implies (iv). The statement is proved by a routine argument.

(iv) implies (v). Since \succeq is probabilistically sophisticated, by Lemma 1, $\text{dom } G \subseteq \mathbb{R} \times \Delta^\sigma(q)$, and by Lemma 2, I is rearrangement invariant. By Theorem 3, (v) holds.

By Theorem 3, (v) implies (ii), which concludes the proof. ■

A.3 Proofs of Corollary 1, Proposition 2, and Proposition 3

Let $I : B_0 \rightarrow \mathbb{R}$ be defined as in (23). For each $\varphi \in B_0$ the normalized Greenberg-Pierskalla superdifferential of I at φ is the set

$$\partial I(\varphi) = \left\{ p \in \Delta : \int \varphi dp \geq \int \psi dp \Rightarrow I(\varphi) \geq I(\psi) \right\}.$$

Proposition 4 Let (u, G) be an uncertainty averse representation for \succsim . The following conditions are equivalent for $\bar{p} \in \Delta$:

- (i) \succsim is more uncertainty averse than $\succsim_{u, \bar{p}}$;
- (ii) $G(t, \bar{p}) = \min_{p \in \Delta} G(t, p)$ for all $t \in \mathbb{R}$;
- (iii) $\bar{p} \in \bigcap_{t \in \mathbb{R}} \partial I(t)$.

Proof. (i) implies (ii). Since $\succsim_{u, \bar{p}}$ is a SEU preference, setting

$$\bar{G}(t, p) = \begin{cases} t & \text{if } (t, p) = (t, \bar{p}) \\ \infty & \text{otherwise} \end{cases} \quad (38)$$

for all $(t, p) \in \mathbb{R} \times \Delta$, it follows that the pair (u, \bar{G}) is an uncertainty averse representation of $\succsim_{u, \bar{p}}$. By [2, Proposition 6], the fact that \succsim is more uncertainty averse than $\succsim_{u, \bar{p}}$ translates into

$$t \leq G(t, p) \leq \bar{G}(t, p) \quad \forall (t, p) \in \mathbb{R} \times \Delta. \quad (39)$$

Substituting $p = \bar{p}$ in (39) delivers

$$G(t, \bar{p}) = t = \min_{p \in \Delta} G(t, p) \quad \forall t \in \mathbb{R}$$

where the last equality follows from $G \in \mathcal{L}(\mathbb{R} \times \Delta)$.

(ii) implies (iii). Since (u, G) is an unbounded uncertainty averse representation,

$$G(t, p) = \sup \left\{ u(x_f) : \int u(f) dp \leq t \right\} = \sup \left\{ I(u(f)) : \int u(f) dp \leq t \right\} = \sup \left\{ I(\psi) : \int \psi dp \leq t \right\} \quad (40)$$

for all $(t, p) \in \mathbb{R} \times \Delta$. Let $\psi \in B_0$ be such that $\int \psi d\bar{p} \leq t$, then, by (40), $I(\psi) \leq G(t, \bar{p})$. But (ii) implies that $G(t, \bar{p}) = t$. Thus $I(\psi) \leq t = I(t)$ implies that if $\int \psi dp \leq \int t d\bar{p}$ then $I(\psi) \leq I(t)$ and we conclude that $\bar{p} \in \partial I(t)$.

(iii) implies (i). Let $f \in \mathcal{F}$ and $x \in X$. Since $\bar{p} \in \bigcap_{t \in \mathbb{R}} \partial I(t)$, it follows that $\bar{p} \in \partial I(u(x))$. Therefore,

$$\int u(f) d\bar{p} \leq \int u(x) d\bar{p} \Rightarrow I(u(f)) \leq I(u(x))$$

that is

$$x \succsim_{u, \bar{p}} f \Rightarrow x \succsim_{u, \bar{p}} u(f).$$

But the latter condition can be easily seen to be equivalent to \succsim being more uncertainty averse than $\succsim_{u, \bar{p}}$.¹³ ■

¹³Indeed, consider two preorders \succsim_1 and \succsim_2 on \mathcal{F} . Assume that \succsim_i over X is represented by an affine nonconstant function u_i and for each $f \in \mathcal{F}$ there exists $x_f^i \sim_i f$ ($i = 1, 2$). Then, the following conditions are equivalent:

- (i) For each $f \in \mathcal{F}$ and each $x \in X$, $f \succsim_1 x \Rightarrow f \succsim_2 x$;
- (ii) \succsim_1 coincides with \succsim_2 on X and $x_f^2 \succsim_1 x_f^1$ for all $f \in \mathcal{F}$;
- (iii) For each $f \in \mathcal{F}$ and each $x \in X$, $f \succ_1 x \Rightarrow f \succ_2 x$ (i.e. $x \succsim_2 f \Rightarrow x \succsim_1 f$).

Proof of Corollary 1. Assume (i). For each $p \in \Delta^\sigma(q)$, by Jensen's inequality,

$$\int \ell \left(\frac{dp}{dq} \right) dq \geq \ell \left(\int \frac{dp}{dq} dq \right) = \ell(1) = \int \ell \left(\frac{dq}{dq} \right) dq$$

for all convex functions $\ell : \mathbb{R} \rightarrow \mathbb{R}$. This implies that $p \succsim_{cx} q$ for all $p \in \Delta^\sigma(q)$. By Theorem 2, $G(t, \cdot)$ is Shur convex for all $t \in \mathbb{R}$. Therefore, for each $t \in \mathbb{R}$, $G(t, p) \geq G(t, q)$ if $p \in \Delta^\sigma(q)$ and $G(t, p) = \infty$ if $p \notin \Delta^\sigma(q)$. That is, $q \in \arg \min G(t, \cdot)$ for all $t \in \mathbb{R}$. By Proposition 4, the statement follows.

Assume (ii). By Theorem 2, (i) is satisfied and the statement follows. \blacksquare

Let $u : X \rightarrow \mathbb{R}$ be a nonconstant affine function and C a subset of Δ . Set

$$f \succsim^* g \iff \int u(f) dp \geq \int u(g) dp \quad \forall p \in C. \quad (41)$$

Notice that \succsim^* is complete (and represented by u) on X , hence the definition of nontrivial unambiguous event can be naturally extended to this more general setting.¹⁴ Next we prove that the equivalence among points (i)-(iii) of Proposition 2 holds more in general for any relation \succsim^* defined as above (the special case is obtained by setting $C = \text{dom}_\Delta G$ and observing that \succsim and \succsim^* coincide on X).

Proof of Proposition 2. (i) implies (iii). By assumption, there exist $x, y, z \in X$ such that $x \succ^* z \succ^* y$ and $xAy \sim^* z$. Wlog $u(x) = 1$ and $u(y) = 0$. Then $x \succ^* z \succ^* y$ amounts to $u(z) \in (0, 1)$ and $xAy \sim^* z$ amounts to $\int u(xAy) dp = u(z)$ for all $p \in C$, that is $p(A) = u(z)$ for all $p \in C$.

(iii) implies (ii). Consider $x, y \in X$ such that $x \succ^* y$. Wlog $u(x) = 1$ and $u(y) = 0$. By (iii), it follows that there exists $\alpha \in (0, 1)$ such that $\int u(xAy) dp = p(A) = \alpha$ for all $p \in C$. Set $z = \alpha x + (1 - \alpha)y$ and conclude $x \succ^* z \succ^* y$ and $xAy \sim^* z$.

(iii) implies (ii). Follows from the fact that \succsim^* is nontrivial on X . \blacksquare

A subset C of $\Delta^\sigma(q)$ is Shur convex (wrt $q \in \Delta^\sigma$) if and only if $\{p \in \Delta^\sigma(q) : p \precsim_{cx} p'\} \subseteq C$ for each $p' \in C$.

Proposition 5 *Let $q \in \Delta^\sigma$ and \succsim^* be defined as in (41). If A is a nontrivial unambiguous event for \succsim^* and C is Schur convex (wrt q), then $C = \{q\}$.*

Proof. Wlog $p(A) = \alpha \in (0, 1/2]$ for all $p \in C$. Let $\bar{p} \in C$, then, by Step 1 of the proof of Theorem 3,

$$\begin{aligned} \max \left\{ \int 1_A dp : \Delta^\sigma(q) \ni p \precsim_{cx} \bar{p} \right\} &= \int_0^1 \delta_{1_A}(\omega) \delta_{\frac{d\bar{p}}{dq}}(\omega) d\omega \\ \min \left\{ \int 1_A dp : \Delta^\sigma(q) \ni p \precsim_{cx} \bar{p} \right\} &= \int_0^1 \delta_{1_A}(\omega) \delta_{\frac{d\bar{p}}{dq}}(1 - \omega) d\omega. \end{aligned}$$

Since $q \in \{p \in \Delta^\sigma(q) : p \precsim_{cx} \bar{p}\} \subseteq C$ and A is a nontrivial unambiguous event, then $\{\int 1_A dp : \Delta^\sigma(q) \ni p \precsim_{cx} \bar{p}\} = \{q(A)\} = \{\alpha\}$, hence,

$$\int_0^1 \delta_{1_A}(\omega) \delta_{\frac{d\bar{p}}{dq}}(\omega) d\omega = \int_0^1 \delta_{1_A}(\omega) \delta_{\frac{d\bar{p}}{dq}}(1 - \omega) d\omega.$$

As well known,

$$\delta_{1_A}(\omega) = \begin{cases} 1 & \omega \in (0, \alpha) \\ 0 & \omega \in [\alpha, 1). \end{cases}$$

¹⁴ An event A in Σ is nontrivial and unambiguous if there exist $x, y, z \in X$ such that $x \succ^* z \succ^* y$ and $xAy \sim^* z$.

Therefore,

$$\int_0^\alpha \delta_{\frac{d\bar{p}}{dq}}(\omega) d\omega = \int_0^\alpha \delta_{\frac{d\bar{p}}{dq}}(1-\omega) d\omega \quad (42)$$

but $\delta_{\frac{d\bar{p}}{dq}} : (0, 1) \rightarrow [0, \infty)$ is decreasing and $\int_0^1 \delta_{\frac{d\bar{p}}{dq}}(\omega) d\omega = 1$ [7, 4.3]. Therefore, by standard arguments, (42) implies $\delta_{\frac{d\bar{p}}{dq}} = 1$ (λ -a.e.). It follows that $\frac{d\bar{p}}{dq} = 1$ (q -a.e.) [7, 2.8], and $\bar{p} = q$. ■

Proof of Proposition 3. Let A be a nontrivial unambiguous event for \succsim .

(i) Sufficiency is immediate. As to necessity, notice that \succsim^* is represented by (18) and $\text{dom}_\Delta G$ is Shur convex subset of $\Delta^\sigma(q)$. Proposition 5 delivers $\text{dom}_\Delta G = \{q\}$. By definition of $\text{dom}_\Delta G$ and since $G \in \mathcal{L}(\mathbb{R} \times \Delta)$, it follows that for each $(t, p) \in \mathbb{R} \times \Delta$

$$G(t, p) = \begin{cases} t & \text{if } (t, p) = (t, q) \\ \infty & \text{otherwise.} \end{cases}$$

The statement follows.

(ii) Assume that (S, Σ, q) is adequate. Sufficiency is trivial. As for necessity, notice that, by Theorem 2, \succsim satisfies second order stochastic dominance and the statement follows. ■

References

- [1] F. J. Anscombe and R. J. Aumann, A definition of subjective probability, *Annals of Mathematical Statistics*, 34, 199–205, 1963.
- [2] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Uncertainty averse preferences, Carlo Alberto WP 77, 2008.
- [3] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Complete quasiconcave monotone duality, Carlo Alberto WP 80, 2008.
- [4] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Risk measures: rationality and diversification, Carlo Alberto WP 100, 2008.
- [5] S. H. Chew and J. S. Sagi, Event exchangeability: Probabilistic sophistication without continuity or monotonicity, *Econometrica*, 74, 771–786, 2006.
- [6] S. H. Chew and J. S. Sagi, Small worlds: Modeling attitudes toward sources of uncertainty, *Journal of Economic Theory*, 139, 1–24, 2008.
- [7] K. M. Chong and N. M. Rice, Equimeasurable rearrangements of functions, *Queen's Papers in Pure and Applied Mathematics*, 28, 1971.
- [8] P. Ghirardato and M. Marinacci, Ambiguity made precise: A comparative foundation, *Journal of Economic Theory*, 102, 251–289, 2002.
- [9] P. Ghirardato, F. Maccheroni, and M. Marinacci, Differentiating ambiguity and ambiguity attitude, *Journal of Economic Theory*, 118, 133–173, 2004.
- [10] I. Gilboa and D. Schmeidler, Maxmin expected utility with a non-unique prior, *Journal of Mathematical Economics*, 18, 141–153, 1989.
- [11] S. Grant, Subjective probability without monotonicity: Or how Machina's mom may also be probabilistically sophisticated, *Econometrica*, 63, 159–189, 1995.

- [12] S. Grant and B. Polak, Bayesian beliefs with stochastic monotonicity: An extension of Machina and Schmeidler, *Journal of Economic Theory*, 130, 264–282, 2006.
- [13] I. Kopylov, Subjective probabilities on “small” domains, *Journal of Economic Theory*, 133, 236–265, 2007.
- [14] W. A. J. Luxemburg, Rearrangement-invariant Banach function spaces, *Queens Papers in Pure and Applied Mathematics*, 10, 83–144, 1967.
- [15] M. J. Machina, D. Schmeidler, A more robust definition of subjective probability, *Econometrica*, 60, 745–780, 1992.
- [16] M. J. Machina, D. Schmeidler, Bayes without Bernoulli: Simple conditions for probabilistically sophisticated choice, *Journal of Economic Theory*, 67, 106–128, 1995.
- [17] M. Marinacci, Probabilistic sophistication and multiple priors, *Econometrica*, 70, 755–764, 2002.
- [18] F. Maccheroni, M. Marinacci, and A. Rustichini, Ambiguity aversion, robustness, and the variational representation of preferences, *Econometrica*, 74, 1447–1498, 2006.
- [19] W. Neilson, Ambiguity aversion: An axiomatic approach using second order probabilities, mimeo, 1993.
- [20] R. Sarin and P. P. Wakker, Cumulative dominance and probabilistic sophistication, *Mathematical Social Sciences*, 40, 191–196, 2000.
- [21] L. J. Savage, *The foundations of statistics*, Wiley, New York, 1954.
- [22] T. Strzalecki, Axiomatic foundations of multiplier preferences, mimeo, 2007.
- [23] T. Strzalecki, Probabilistic sophistication and variational preferences, mimeo, 2008.
- [24] P. Wakker, Clarification of some mathematical misunderstandings about Savage’s foundations of statistics 1954, *Mathematical Social Sciences*, 25, 199–202, 1993.